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Surfactant induced one dimensional unsaturated flow through porous media: a classical mathematical approach

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Abstract Surfactants reduce the interfacial tension, amend the solid–liquid contact angle and greatly influence the capillarity action in unsaturated porous media. Solubility studies of surfactants in inducing similar flow through such medium has been described to be of great importance to hydrologists, agriculturists and for the people related with water sciences to confine the flow problems in water infiltration system, seepage delinquent and the underground disposal of wastewater. Present article reviews the current state of knowledge to understand such one dimensional, unsteady surfactant flow phenomenon due to the capillary pressure gradients and is represented mathematically using one parameter group theory of similarity analysis. For the sake of definiteness in the analysis, we assumed certain specific relationships *viz.* the permeability of the medium as a specific linear function of moisture content and time which are consistent with the physical problem. We have not included any graphical or numerical illustrations due to our particular interest in deriving the classical solution to our problem.

Keywords Surfactant · Surface tension · Porous media · Unsaturated flow · Analytical solution · One parameter group theory

1 Introduction

Organic compounds, either aliphatic or aromatic, water soluble or insoluble can act and referred as surface active agent (*Surfactant*) when found enough capable to form a

Analytical methodology in terms of confluent hypergeometric functions of the non-linear partial differential equation governing one dimensional surfactant induced unsaturated flow through porous media.

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uniform condensed film at their equilibrium spreading [1–5]. Such self-association and well-defined orientation at the air/water interface with sharp polarity gradient directs the novel ability of the surfactants to undergo an efficient close packing resulting into an effective lowering of surface tension by several times [6–8]. It is commonly known that the surface tension of most of the organic compounds is lower than pure water. Such behavior could be attributed due to the relative aqueous-phase flow concentration and to the degree of hydrophobicity [3,9–11]. However the magnitude of its effect is also predicted to be system-specific and depends on the degree of retention variation between the surfactant-free and surfactant-contaminated regions, the characteristics of the porous medium and the wetting–drying history of the investigated system. As surfactant-induced reduction in surface tension is directly proportional to capillary pressure; consequently it prompts the capillary pressure gradients which could be sufficient to cause the unsaturated flow perturbations [11]. All these factors count on the surfactant type and concentration. Due to such unique surface active nature and solution conduct these systems have been hired for various commercial industrial applications, in engineered microfluidic systems and subsurface remedial schemes [12,13].

Different *mathematical* viewpoints, approaches, variable techniques and approximate solutions are putforth for better understanding on such unsaturated aqueous flow [14–17]. Granting such one dimensional unsaturated flow through porous media induced by the capillary pressure gradient caused by solute (*surfactant*) concentration; we presented in particular the state of knowledge on water-wetted porous media [18,19] and have attempted to derive mathematically an analytical solution under certain assumptions and obtain its pure classical solution. Here the partial non-linear differential equation associated with a flow is transformed into an ordinary differential equation using one parameter group theory of similarity analysis. The scope of the paper is limited to the experimental data for the physico-chemical characterization of surfactants in aqueous medium as our primary interest aimed in deriving the classical solution to our problem.

2 Mathematical formulation of the problem

Considering, the motion of surfactant solution flow through unsaturated porous media is governed by the continuity equation, we have

$$\frac{\partial(\rho_s.u)}{\partial t} = \nabla.M \tag{1}$$

where ρ_s is the bulk density of medium on dry weight basis, u is the moisture content at any depth z on a dry weight basis and M is the mass flux of moisture [19].

From Darcy's law for the motion of water in a porous medium, we have

$$V = -k\nabla_{\emptyset} \tag{2}$$

where V is the volume flux of moisture, k is the coefficient of aqueous conductivity and $\nabla_{\emptyset}$ is the gradient of the whole moisture potential.

Combining equations (1) and (2), we get

$$\frac{\partial(\rho_s.u)}{\partial t} = \nabla(\rho k \nabla_{\emptyset}), \text{ where } \rho \text{ is the flux density} \tag{3}$$

Since the flow takes place only in vertical direction, Eq. (3) reduces to

$$\rho_s \frac{\partial u}{\partial t} = \frac{\partial}{\partial z}\left(\rho k \frac{\partial \Psi}{\partial z}\right) + \frac{\partial}{\partial z}(\rho k g) \tag{4}$$

where Ψ is pressure (capillary) potential, g is the gravitational constant and $\emptyset = \Psi - gz$. The positive direction of z-axis is the same as that of the gravity.

Considering u and Ψ to be connected by a single valued function, Eq. (4) can be written as

$$\therefore \frac{\partial u}{\partial t} = \frac{\partial}{\partial z}\left\{D\frac{\partial u}{\partial z}\right\} + \frac{\rho_g}{\rho_s}\frac{\partial k}{\partial z},$$

$$\text{where } D = \frac{\rho_k}{\rho_s}\frac{\partial \Psi}{\partial u} \text{ which is called the } diffusivity\, coefficient \tag{5}$$

It is assumed that the diffusivity coefficient D is equivalent to the average value D_a over the whole range of the moisture content, the permeability k of the medium is considered to vary directly with the moisture content u and inversely as the square root of time t. So therefore the Eq. (5) becomes

$$\frac{\partial u}{\partial t} = D_a \frac{\partial^2 u}{\partial^2 z} + \frac{\rho_g}{\rho_s}\frac{k_0}{\sqrt{t}}\frac{\partial u}{\partial z}, \quad \text{where } k = \frac{k_0}{\sqrt{t}}u; \quad (k_0 = 0.232) \tag{6}$$

For definiteness of the physical problem, we consider the downward and vertical water flow will obey the following boundary conditions:

$$u(0,t) = u_0;\ u(L,t) = u_L(\text{ where } u_L \neq 1) \text{ and } \frac{\rho_g}{\rho_s}\frac{k_0}{\sqrt{t}} = \frac{u}{L} \tag{7}$$

3 Analytical solution

This section discusses the solution of the above boundary value problem (6) which is transformed into ordinary differential equation by using similarity variable.

Using the dimensionless variables, $Z = \frac{z}{L}$ and $T = \frac{t}{L^2}$ in (6), we have the following solution to the boundary value problem for the mentioned flow as

$$\frac{\partial u}{\partial T} = D_a \frac{\partial^2 u}{\partial Z^2} + u\frac{\partial u}{\partial Z} \tag{8}$$

Let the solution of the problem (8) be given by

$$u(Z,T)=G(Z,T).H(T)$$
$$\text{i.e. } u=GH \tag{9}$$

Now $\frac{\partial u}{\partial T}=H\frac{\partial G}{\partial T}+G\frac{\partial H}{\partial T}; \frac{\partial u}{\partial Z}=\frac{\partial}{\partial Z}(GH)=H\frac{\partial G}{\partial Z}$

$$\frac{\partial^2 u}{\partial Z^2}=\frac{\partial}{\partial Z}\left(H\frac{\partial G}{\partial Z}\right)=H\frac{\partial^2 G}{\partial Z^2}$$

Thus differential equation (8) takes the form

$$H\frac{\partial G}{\partial T}+G\frac{\partial H}{\partial T}=D_a H\frac{\partial^2 G}{\partial Z^2}+GH^2\frac{\partial G}{\partial Z} \tag{10}$$

Hence, the boundary conditions are transformed into:

$$u_0=(0,T)\,H(T) \text{ and } u_L=G\,(L,T)\,H\,(T) \tag{11}$$

3.1 One parameter group transformations

The procedure is initiated with the following group G, a class of transformations of one-parameter 'a' of the form:

$$\begin{cases} \text{G}: & \bar{Z}=\mathbb{C}^Z(a)\,Z+K^Z(a), \quad \bar{H}=\mathbb{C}^H(a)\,H+K^H(a) \\ & \bar{T}=\mathbb{C}^T(a)\,T+K^T(a), \quad \bar{G}=\mathbb{C}^G(a)\,G+K^G(a) \end{cases} \tag{12}$$

where $\mathbb{C}'s$ and $K's$ are real-valued differentiable functions in the real parameter 'a'.

From Eq. (12), we have

$$\frac{\partial \bar{Z}}{\partial Z}=\mathbb{C}^Z(a); \quad \frac{\partial \bar{T}}{\partial T}=\mathbb{C}^T(a); \frac{\partial \bar{H}}{\partial H}=\mathbb{C}^H(a); \frac{\partial \bar{G}}{\partial G}=\mathbb{C}^G(a)$$

Now,

$$\frac{\partial \bar{G}}{\partial \bar{T}}=\frac{\partial \bar{G}}{\partial G}\cdot\frac{\partial G}{\partial \bar{T}}=\frac{\partial \bar{G}}{\partial G}\frac{\partial G}{\partial T}\frac{\partial T}{\partial \bar{T}}=\frac{\mathbb{C}^G(a)}{\mathbb{C}^T(a)}\frac{\partial G}{\partial T},$$
$$\frac{\partial \bar{H}}{\partial \bar{T}}=\frac{\partial \bar{H}}{\partial H}\cdot\frac{\partial H}{\partial \bar{T}}=\frac{\partial \bar{H}}{\partial H}\frac{\partial H}{\partial T}\frac{\partial T}{\partial \bar{T}}=\frac{\mathbb{C}^H(a)}{\mathbb{C}^T(a)}\frac{\partial H}{\partial T} \text{ and}$$
$$\frac{\partial \bar{G}}{\partial \bar{Z}}=\frac{\partial \bar{G}}{\partial G}\cdot\frac{\partial G}{\partial \bar{Z}}=\frac{\partial \bar{G}}{\partial G}\frac{\partial G}{\partial Z}\frac{\partial Z}{\partial \bar{Z}}=\frac{\mathbb{C}^G(a)}{\mathbb{C}^Z(a)}\frac{\partial G}{\partial Z}$$
$$\frac{\partial^2 \bar{G}}{\partial \bar{Z}^2}=\frac{\partial}{\partial \bar{Z}}\left(\frac{\partial \bar{G}}{\partial \bar{Z}}\right)=\frac{\partial}{\partial \bar{Z}}\left(\mathbb{C}^G(a)\frac{\partial G}{\partial Z}\frac{1}{\mathbb{C}^Z(a)}\right)=\frac{\mathbb{C}^G(a)}{\mathbb{C}^Z(a)}\cdot\frac{\partial}{\partial Z}\left(\frac{\partial G}{\partial Z}\right)\frac{\partial Z}{\partial \bar{Z}}=\frac{\mathbb{C}^G(a)}{[\mathbb{C}^Z(a)]^2}\frac{\partial^2 G}{\partial Z^2} \tag{13}$$

Equation (10) is said to be invariantly transformed for some function $M(a)$, whenever

$$\bar{H}\frac{\partial \bar{G}}{\partial \bar{T}} + \bar{G}\frac{\partial \bar{H}}{\partial \bar{T}} - \bar{H}D_a\frac{\partial^2 \bar{G}}{\partial \bar{Z}^2} - \bar{G}(\bar{H})^2\frac{\partial \bar{G}}{\partial \bar{Z}} = M(a)\left[H\frac{\partial G}{\partial T} + G\frac{\partial H}{\partial T} - HD_a\frac{\partial^2 G}{\partial Z^2} - GH^2\frac{\partial G}{\partial Z}\right] \quad (14)$$

Substituting Eq. (13) in (14) and on further solving, we get

$$\left[\frac{\mathbb{C}^H\mathbb{C}^G}{\mathbb{C}^T}\right]H\frac{\partial G}{\partial T} + \left[\frac{\mathbb{C}^H\mathbb{C}^G}{\mathbb{C}^T}\right]G\frac{\partial H}{\partial T} - \left[\frac{\mathbb{C}^H\mathbb{C}^G}{(\mathbb{C}^Z)^2}\right]D_aH\frac{\partial^2 G}{\partial Z^2} - \left[\frac{\left(\mathbb{C}^H\mathbb{C}^G\right)^2}{\mathbb{C}^Z}\right] GH^2\frac{\partial G}{\partial Z} + R(a) = M(a)\left[H\frac{\partial G}{\partial T} + G\frac{\partial H}{\partial T} - D_aH\frac{\partial^2 G}{\partial Z^2} - GH^2\frac{\partial G}{\partial Z}\right] \quad (15)$$

where $R(a) = \frac{K^H\mathbb{C}^G}{\mathbb{C}^T}\frac{\partial G}{\partial T} + \frac{K^H\mathbb{C}^H}{\mathbb{C}^T}\frac{\partial H}{\partial T} - \frac{K^H\mathbb{C}^G}{(\mathbb{C}^Z)^2}D_a\frac{\partial^2 G}{\partial Z^2} - \frac{\left(\mathbb{C}^G\right)^2}{\mathbb{C}^Z}G^2\mathbb{C}^H HK^H\frac{\partial G}{\partial Z} - \frac{\left(\mathbb{C}^G\right)^2}{\mathbb{C}^Z}G(K^H)^2\frac{\partial G}{\partial Z} - K^G(\mathbb{C}^H H)^2\frac{\mathbb{C}^G}{\mathbb{C}^Z}\frac{\partial G}{\partial Z} - 2\mathbb{C}^H HK^GK^H\frac{\mathbb{C}^G}{\mathbb{C}^Z}\frac{\partial G}{\partial Z} - K^G(K^H)^2\frac{\mathbb{C}^G}{\mathbb{C}^Z}\frac{\partial G}{\partial Z}$

The invariance of (15) implies $R(a) = 0$ which is satisfied by putting $K^H = K^G = 0$

Hence, we get $\frac{\mathbb{C}^H\mathbb{C}^G}{\mathbb{C}^T} = \frac{\mathbb{C}^H\mathbb{C}^G}{(\mathbb{C}^Z)^2} = \frac{(\mathbb{C}^H\mathbb{C}^G)^2}{\mathbb{C}^Z} = M(a)$ which yields

$$\mathbb{C}^T = (\mathbb{C}^Z)^2, \quad \frac{1}{\mathbb{C}^Z} = \mathbb{C}^G\mathbb{C}^H \quad (16)$$

Moreover, boundary conditions of Eq. (7) are also invariant in form, implying that

$$\bar{u}(0, \bar{T}) = u_0 \text{ and } \bar{u}(L, \bar{T}) = u_L$$

Now,

$$\bar{Z} = \mathbb{C}^Z(a)\,Z + K^Z(a)$$

whenever $Z = 0$ then $\bar{Z} = \mathbb{C}^Z(a)\cdot 0 + K^Z(a) = K^Z(a)$

But we require $\bar{Z} = 0$ which is possible if $K^Z(a) = 0$

$$\therefore K^Z = 0 \quad (17)$$

Now,

$$\bar{T} = \mathbb{C}^T(a)\,T + K^T(a)$$

whenever $T = 0$ then $\bar{T} = \mathbb{C}^T(a)\cdot 0 + K^T(a) = K^T(a)$

But we require $\bar{T} = 0$ and this is possible if $K^T(a) = 0$

$$\therefore K^T = 0 \tag{18}$$

Finally, we get the one-parameter group G which transforms invariantly differential equation (15) as well as boundary conditions (17) and (18)

Thus, the group G (12) is of the form

$$\begin{cases} \text{G}: & \bar{Z} = \mathbb{C}^Z Z, \quad \bar{H} = H \\ & \bar{T} = (\mathbb{C}^Z)^2 T, \quad \bar{G} = G \end{cases} \tag{12a}$$

Thus aiming to use the group theory method to represent the problem in the form of an ordinary differential equation, we precede our analysis to obtain a complete set of absolute invariants.

If $\eta = \eta(Z, T)$ is the absolute invariant of the independent variables, then

$$g_j(Z, T; G, H) = F_j[\eta(Z, T)] \qquad \text{where } j = 1, 2 \tag{19}$$

are two absolute invariants corresponding to G and H.

From (17), $\eta(Z, T)$ is an absolute invariant if it satisfies

$$\alpha_1 Z \frac{\partial \eta}{\partial Z} + \alpha_2 T \frac{\partial \eta}{\partial T} = \eta \tag{20}$$

The characteristics equations are

$$\begin{aligned} & \frac{dZ}{\alpha_1 Z} = \frac{dT}{\alpha_2 T} = \eta \\ & \frac{1}{\alpha_1} log Z = \frac{1}{\alpha_2} log T \text{ (on integrating)} \\ & Z = T^{\frac{\alpha_1}{\alpha_2}} \\ & \therefore \eta(Z, T) = \frac{Z}{T^\beta}, \text{ where } \beta = \frac{\alpha_1}{\alpha_2} > 0, \text{ is the solution} \end{aligned} \tag{21}$$

By similarity analysis the absolute invariants of the dependent variables u and q are

$$H(T) = q(T)\Psi(\eta) \tag{22}$$

Since $q(T)$ and $H(T)$ are independent of Z, while η is a function of Z and T, then $\Psi(\eta) = 1$

$$\therefore H(T) = q(T) \tag{23}$$

The second absolute invariant is

$$G(Z, T) = F(\eta) \tag{24}$$

Here $\eta = \frac{Z}{T^\beta}$ and $H(T) = q(T)$

3.2 Reduction into an ordinary differential equation

Substituting (22), (23) and (24) into Eq. (10), we have

$$\therefore F''(\eta) + (T^{\beta} F q + T^{\beta-1})F'(\eta) - \frac{1}{q}T^{2\beta} q' F = 0 \tag{25}$$

For (25) to be reduced to an expression in the single independent invariant η, it is necessary that the coefficients should be constants or functions of η alone. So we consider the constant coefficients only.

$$Z\beta T^{\beta-1} = K_1 \tag{26}$$

$$T^{\beta} G H = K_2 \tag{27}$$

$$\frac{1}{q}T^{2\beta} q' = K_3 \tag{28}$$

Taking $\beta = 0.5$, $K_3 = -0.5$ and on solving Eq. (28), we get

$$q = T^{-1/2} = \frac{1}{\sqrt{T}}$$

Substituting the above values in Eq. (25), we get the final reduced form of ordinary differential equation

$$2F''(\eta) + (2F + \eta)F'(\eta) + F = 0 \tag{29}$$

Under the similarity variable η, the boundary conditions (29) are transformed into:

$$F(0) = \frac{u_0}{H(T)} \text{ and F}\left(\frac{1}{T^{\beta}}\right) = \frac{u_L}{H(T)}$$

Thus, the Eq. (29) takes the form

$$\frac{d}{d\eta}\left(\frac{dF}{d\eta} + \frac{1}{2}F^2 + \frac{1}{2}\eta F\right) = 0 \text{ which on integration gives}$$

$$\frac{dF}{d\eta} + \frac{1}{2}F^2 + \frac{1}{2}\eta F = K_1, \text{ where } K_1 \text{ is constant}$$

$$\text{i.e. } F'(\eta) = K_1 - \frac{1}{2}\eta F - \frac{1}{2}F^2 \tag{30}$$

Substitute, $-\frac{1}{2}F(\eta)u + u' = 0$ and differentiating w.r.t 'η', we get

$$u''(\eta) - g u'(\eta) + a f u(\eta) = 0 \tag{31}$$

Putting, $g = -\frac{1}{2}\eta, a = -\frac{1}{2}, f = K_1$

$$\therefore u''(\eta) + \frac{1}{2}\eta u'(\eta) + \left(-\frac{1}{2}K_1\right) u = 0 \tag{32}$$

Substituting, $u(\eta) = v(z)$, where $2z = -a\eta^2$, we get the confluent hypergeometric differential equation

$$zv''(z) + \left(\frac{1}{2} - z\right) v' - \frac{1}{2}K_1 v = 0 \tag{33}$$

It is also called the *Kummer equation* and the *Pochhammer–Barnes equation* and a solution of it is a confluent hypergeometric function, a *Kummer function*, or a *Pochhammer function*.

The general solution of Eq. (33) is

$$\begin{aligned} v &= c_1 v_1 + c_2 v_2 \\ v_1 &= 1^F 1(a, c; z) \\ v_2 &= Z^{1-c} 1^F 1(1 + a - c, 2 - c; z) \end{aligned} \tag{34}$$

where $1^F 1(a, c; z) = \sum_{k=0}^{\infty} A_k Z^k$; where $A_k = \frac{a(a+1)\ldots(a+k-1)}{c(c+1)\ldots(c+k-1)k!}$ and

$$Z^{1-c} 1^F 1(1 + a - c, 2 - c; z) = \sum_{k=0}^{\infty} B_k Z^k; \text{ where}$$
$$B_k = \frac{(1 + a - c)(2 + a - c)\ldots(a + k - c)}{(2 - c)(3 - c)\ldots(1 - c + k)\, k!}$$

If $a - c = \frac{1}{2}K_1 - \frac{1}{2} = \frac{1}{2}(K_1 - 1) = n$ (integer) where $K_1 = 2n + 1(K_1$ is odd), then v_1 becomes $e^z 1^F 1(-n, c; -z)$ and the series has become a polynomial. The other solution is an infinite series.

Similarly, if $a - 1 = n \Rightarrow \frac{1}{2}K_1 - 1 = n \Rightarrow K_1 = 2n + 1$, the polynomial solution is $v_2 = Z^{1-c} e^z 1^F 1(-n, 2 - c; -z)$.

4 Conclusion

Thus using one parameter group theory of similarity analysis, this article presents the analytical solution in terms of confluent hypergeometric functions of the nonlinear partial differential equation which is transformed into an ordinary differential equation for one dimensional, unsteady surfactant flow through unsaturated porous media. For the sake of assurance in the analysis, the present discussion assumes certain specific relationships which are consistent and holds equally true to the present physical hydrological problems. Future research directions should include the collection of additional field and laboratory-scale data and expanded modeling efforts.

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